

Q -curvature type problem on bounded domains of \mathbb{R}^n

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Abstract. In this paper, we establish compactness and existence results to a Branson-Paneitz type problem on a bounded domain of \mathbb{R}^n with Navier boundary condition.

MSC 2000: 35J60, 35J60, 58E05.

Key words: Nonlinear elliptic P D E, critical exponent, Lack of compactness, Critical points at infinity.

1 Introduction

In this work we are concerned with positive solutions of a nonlinear fourth order equation under the Navier boundary condition. Let K be a given function on a smooth bounded domain Ω of $\mathbb{R}^n, n \geq 5$. We are looking for a map $u : \Omega \rightarrow \mathbb{R}$ satisfying the following critical fourth order PDE

$$\begin{cases} \Delta^2 u = K(x) u^{\frac{n+4}{n-4}}, \\ u > 0 \text{ in } \Omega, \\ \Delta u = u = 0 \text{ on } \partial\Omega. \end{cases} \quad (1.1)$$

The interest of this equation comes from its resemblance to the so called Q -curvature problem on closed manifolds involving the Branson-Paneitz operator. The latter has widely studied in the last two decades. (See [2] [12], [6], [7], [3], [10], [11], [15], [16], [17] and the references therein for details).

Problem (1.1) has a variational structure with challenging mathematical difficulties. Indeed, if there is general standard line of attack to solve the analogous of (1.1) in the subcritical case. These approaches do not apply to the critical case since the embedding $\mathcal{H} \hookrightarrow L^{\frac{2n}{n-4}}(\Omega)$ where $\mathcal{H} := H^2(\Omega) \cap H_0^1(\Omega)$, is not compact.

When $K = 1$, the problem is called the Yamabe type problem. In this case, the existence of solutions of problem (1.1) depends on the topology of Ω . More precisely, if Ω is a

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star-shaped bounded domain, Van der Vorst [21] proved that (1.1) has no solution. When Ω has a non trivial homology group, Ebobisse-Ould Ahmedou showed that (1.1) has a solution [18].

When $K \neq 1$, there have been many works dealing with (1.1). In these contributions, the conditions on $K(x)$ ensuring the solvability of (1.1) have been discussed. In [6], [13] and [14], some existence results were obtained under the following two hypotheses:

$$(A) \quad \frac{\partial K}{\partial \nu}(x) \neq 0, \quad \forall x \in \partial\Omega.$$

Here ν is the unit outward normal vector on $\partial\Omega$.

(nd) K is a C^2 -positive function having only non degenerate critical points such that

$$\Delta K(x) \neq 0 \text{ if } \nabla K(x) = 0.$$

Observe that (nd)-condition would excludes some interesting class of functions K . For example the C^1 -functions and smooth functions having degenerate critical points. Our main motivation in this study, is to include a wider class of functions K for which (1.1) admits a solution. Our main assumption is the following β -flatness condition:

(f) $_{\beta}$ Assume that K is a C^1 -positive function on $\overline{\Omega}$ such that for each critical point y of K , there exists a real number $\beta = \beta(y) > 1$, such that

$$K(x) = K(y) + \sum_{k=1}^n b_k |(x-y)_k|^{\beta} + R(x-y),$$

for x close to y . Here $b_k = b_k(y) \in \mathbb{R} \setminus \{0\}$, for $k = 1 \dots, n$, and $\sum_{s=0}^{[\beta]} |\nabla^s R(z)| |z|^{s-\beta} = o(1)$, as z tends to zero. Here $[\beta]$ denotes the integer part of β .

Note that the above mentioned (nd)-condition is a particular case of the β -flatness assumption (in a suitable coordinates system) taking $\beta(y) = 2$ for any critical point y of K .

In the first part of this paper, we are interested with the case $1 < \beta \leq n-4, n \geq 6$. Our aim is to provide a full description of the lack of compactness of the associated variational problem to (1.1). Indeed, we will give a characterization of all critical points at infinity of the functional J in Σ^+ and state an Euler-Hopf type of existence result.

Let G denote the Green's function of the bilaplacian under Navier boundary condition on Ω . It is defined by

$$G(x, y) = |x - y|^{-(n-4)} - H(x, y), \text{ for } x \neq y \in \Omega,$$

where H its regular part.

Let \mathcal{K} denote the set consisting of all critical points of $K(x)$. For any $y \in \mathcal{K}$, we define

$$\tilde{i}(y) = \sharp\{b_k(y), b_k(y) < 0\}.$$

Let

$$\mathcal{K}_{<n-4} = \{y \in \mathcal{K}, \beta(y) < n-4\}, \quad \mathcal{K}_{<n-4}^+ = \{y \in \mathcal{K}_{<n-4}, -\sum_{k=1}^n b_k(y) > 0\},$$

$$\mathcal{K}_{n-4} = \{y \in \mathcal{K}, \beta(y) = n-4\}, \quad \mathcal{K}_{n-4}^+ = \{y \in \mathcal{K}_{n-4}, -c_1 \sum_{k=1}^n b_k(y) + c_2 H(y, y) > 0\},$$

and

$$\mathcal{K}_{>n-4} = \{y \in \mathcal{K}, \beta(y) > n-4\},$$

where

$$c_1 = \int_{\mathbb{R}^n} \frac{|x_1|^{n-4}}{(1+|x|^2)^n} dx, \quad c_2 = \int_{\mathbb{R}^n} \frac{dx}{(1+|z|^2)^{\frac{n+4}{2}}}.$$

Here x_1 is the first component of x in some geodesic normal coordinates system.

To any p -tuple of distinct points $\tau_p = (y_{\ell_1}, \dots, y_{\ell_p}) \in \mathcal{K}_{n-4}^p, 1 \leq p \leq \sharp\mathcal{K}$, we associate a $p \times p$ symmetric matrix $M(\tau_p) = (m_{ij})_{1 \leq i, j \leq p}$ defined by:

$$m_{ii} = -\frac{1}{K(y_{\ell_i})^{\frac{n}{4}}} \left(c_1 \sum_{k=1}^n b_k(y_{\ell_i}) - c_2 H(y_{\ell_i}, y_{\ell_i}) \right) \text{ and}$$

$$m_{ij} = -c_2 \frac{G(y_{\ell_i}, y_{\ell_j})}{\left(K(y_{\ell_i}) K(y_{\ell_j}) \right)^{\frac{n-4}{8}}}, \text{ for } i \neq j.$$

Let $\rho(\tau_p)$ be the least eigenvalue of $M(\tau_p)$.

(B) Assume that $\rho(\tau_p) \neq 0$ for any $1 \leq p \leq \sharp\mathcal{K}$.

Lastly define

$$\mathcal{C}_{n-4}^\infty := \{ \tau_p = (y_{l_1}, \dots, y_{l_p}) \in (\mathcal{K}_{n-4}^+)^p, 1 \leq p \leq \sharp\mathcal{K}, \text{ s.t. } y_{\ell_i} \neq y_{\ell_j} \ \forall i \neq j, \text{ and } \rho(\tau_p) > 0 \},$$

$$\mathcal{C}_{<n-4}^\infty := \{ \tau_p = (y_{l_1}, \dots, y_{l_p}) \in (\mathcal{K}_{<n-4}^+)^p, 1 \leq p \leq \sharp\mathcal{K}, \text{ s.t. } y_{\ell_i} \neq y_{\ell_j} \ \forall i \neq j \}$$

and

$$\mathcal{C}^\infty := \mathcal{C}_{n-4}^\infty \cup \mathcal{C}_{<n-4}^\infty \cup \left(\mathcal{C}_{n-4}^\infty \times \mathcal{C}_{<n-4}^\infty \right).$$

The following result describes the lack of compactness of the problem (1.1).

Theorem 1.1 *Under the assumptions (A), (B) and $(f)_\beta$ for $1 < \beta \leq n - 4$. The critical points at infinity of the associated variational problem to (1.1) (see Definition 2.4) are:*

$$(y_{\ell_1}, \dots, y_{\ell_p})_\infty := \sum_{i=1}^p \frac{1}{K(y_{\ell_i})^{\frac{n-4}{2}}} P\delta_{(y_{\ell_i}, \infty)}$$

where $(y_{\ell_1}, \dots, y_{\ell_p}) = (\tau_p) \in \mathcal{C}^\infty$. The index of a such critical points at infinity is $i(\tau_p) = p - 1 - \sum_{i=1}^p (n - \tilde{i}(y_{\ell_i}))$.

The characterization of the critical points at infinity allows us to prove the following existence result.

Theorem 1.2 *Suppose that (A), (B) and $(f)_\beta$ for $1 < \beta \leq n - 4$ hold. If additionally*

$$\sum_{\tau_p \in \mathcal{C}^\infty} (-1)^{i(\tau_p)} \neq 1,$$

then (1.1) has a solution.

In the second part of this paper, we are interested to the case of any $\beta > 1$. We prove a partial description of the lack of compactness of the problem in that case and we provide a perturbation result.

Theorem 1.3 *Assume that K satisfies (A) and $(f)_\beta$ for $\beta > 1$. The critical points at infinity in $V(1, \varepsilon)$ are*

$$P\delta_{(y, \infty)}, y \in \mathcal{K}_{<n-4}^+ \cup \mathcal{K}_{n-4}^+ \cup \mathcal{K}_{>n-4}.$$

Such critical points at infinity has an index equal to $n - \tilde{i}(y)$

Now we state our perturbation result.

Theorem 1.4 *Under the assumptions (A) and $(f)_\beta$ for $\beta > 1$, if*

$$\sum_{y \in \mathcal{K}_{<n-4}^+ \cup \mathcal{K}_{n-4}^+ \cup \mathcal{K}_{>n-4}} (-1)^{n-\tilde{i}(y)} \neq \chi(\Omega)$$

then (1.1) has a solution. Here $\chi(\Omega)$ is the Euler-Poincaré characteristic of Ω .

Our method hinges on the critical points at infinity theory of A. Bahri [4]. In section 2, we state the variational structure associated to problem (1.1). In section 3, we provide an asymptotic expansion of the gradient of J , without assuming any upper bound condition on the β -flatness condition. In section 4, we characterize the critical points at infinity and we prove Theorems 1.1 and 1.3. Lastly in section 5, we prove Theorems 1.2 and 1.4.

2 Preliminaries tools

Let $\mathcal{H} := H^2(\Omega) \cap H_0^1(\Omega)$ with the norm

$$\|u\|_{\mathcal{H}} = \left(\int_{\Omega} (\Delta u(x))^2 dx \right)^{\frac{1}{2}}.$$

Define

$$\Sigma = \{ u \in \mathcal{H}, \text{ s.t. } \|u\|_{\mathcal{H}} = 1 \}, \text{ and } \Sigma^+ = \{ u \in \Sigma, u > 0 \}.$$

Let

$$J(u) = \frac{\left(\int_{\Omega} (\Delta u)^2 \right)^{\frac{n}{n-4}}}{\int_{\Omega} K(x) u^{\frac{2n}{n-4}} dx}.$$

Observe that if u is a critical point of J on Σ^+ , then $J(u)^{\frac{n-4}{8}} \cdot u$ is a solution of (1.1).

J does not satisfy the Palais-Smale condition on Σ^+ (P.S for short). This is due to the loss of compactness of the embedding $\mathcal{H} \hookrightarrow L^{\frac{2n}{n-4}}(\Omega)$. Next, we describe the sequences failing P.S condition. For $a \in \Omega$ and $\lambda > 0$, let

$$\delta_{a,\lambda}(x) = c_n \left(\frac{\lambda}{1 + \lambda^2 |x - a|^2} \right)^{\frac{n-4}{2}}, \quad (2.1)$$

where c_n is a positive constant chosen such that $\delta_{a,\lambda}$ is the family of solutions of the following problem (see [19]):

$$\Delta^2 u = |u|^{\frac{8}{n-4}} u, \quad u > 0 \quad \text{in } \mathbb{R}^n. \quad (2.2)$$

Let $P\delta_{a,\lambda}$ the unique solution of

$$\begin{cases} \Delta^2 P\delta_{a,\lambda} = \delta_{a,\lambda}^{\frac{n+4}{n-4}} & \text{in } \Omega \\ P\delta_{a,\lambda} = \Delta P\delta_{a,\lambda} = 0 & \text{on } \partial\Omega. \end{cases}$$

We have the following estimates where originally introduced by Bahri [4].

$$P\delta_{a,\lambda} = \delta_{a,\lambda} - \frac{c}{\lambda^{\frac{n-4}{2}}} H(a, \cdot) + O\left(\frac{1}{\lambda^{\frac{n}{2}} d^{n-2}}\right), \quad (2.3)$$

$$\lambda \frac{\partial P\delta_{a,\lambda}}{\partial \lambda} = \lambda \frac{\partial \delta_{a,\lambda}}{\partial \lambda} + \frac{n-4}{2} \frac{c}{\lambda^{\frac{n-4}{2}}} H(a, \cdot) + O\left(\frac{1}{\lambda^{\frac{n}{2}} d^{n-2}}\right), \quad (2.4)$$

$$\frac{1}{\lambda} \frac{\partial P\delta_{a,\lambda}}{\partial a_k} = \frac{1}{\lambda} \frac{\partial \delta_{a,\lambda}}{\partial a_k} - \frac{c}{\lambda^{\frac{n-4}{2}}} \frac{\partial H(a, \cdot)}{\partial a_k} + O\left(\frac{1}{\lambda^{\frac{n+2}{2}} d^{n-1}}\right), \quad (2.5)$$

where c is a fixed positive constant, $d = d(a, \partial\Omega)$, a_k is the k^{th} coordinate of a .

We define now the set of potential critical points at infinity associated to J_{ε_0} . Let for $\varepsilon > 0$ and $p \in \mathbb{N}^*$,

$$V(p, \varepsilon) = \left\{ \begin{array}{l} u \in \Sigma^+, s.t., \exists a_1, \dots, a_p \in \Omega, \exists \lambda_1, \dots, \lambda_p > \varepsilon^{-1} \text{ and} \\ \alpha_1, \dots, \alpha_p > 0 \text{ with } \| u - \sum_{i=1}^p \alpha_i P\delta_{a_i, \lambda_i} \| < \varepsilon, \varepsilon_{ij} < \varepsilon \forall i \neq j, \\ \lambda_i d_i > \varepsilon^{-1} \text{ and } |J^{\frac{n}{n-4}}(u) \alpha_i^{\frac{8}{n-4}} K(a_i) - 1| < \varepsilon \forall i = 1, \dots, p. \end{array} \right.$$

Here, $d_i = d(a_i, \partial\Omega)$ and $\varepsilon_{ij} = \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |a_i - a_j|^2 \right)^{\frac{4-n}{2}}$.

Proposition 2.1 ([5], [20]) *Assume that J has no critical points in Σ^+ . Let $(u_k)_k$ be a sequence in Σ^+ such that $J(u_k)$ is bounded and $\partial J(u_k)$ goes to zero. Then there exists a positive integer p , a sequence (ε_k) with $\varepsilon_k \rightarrow 0$ as $k \rightarrow +\infty$ and an extracted subsequence of $(u_k)_k$'s, again denoted $(u_k)_k$, such that $u_k \in V(p, \varepsilon_k), \forall k$.*

The following Proposition gives a parametrization of $V(p, \varepsilon)$.

Proposition 2.2 ([5]) *For all $p \in \mathbb{N}^*$, there exists $\varepsilon_p > 0$ such that for any $\varepsilon \leq \varepsilon_p$ and any u in $V(p, \varepsilon)$, the problem*

$$\min \left\{ \left\| u - \sum_{i=1}^p \alpha_i P\delta_{a_i, \lambda_i} \right\|, \alpha_i > 0, \lambda_i > 0, a_i \in \Omega \right\}.$$

has a unique solution (up to a permutation). Thus, we can uniquely write u as follows

$$u = \sum_{i=1}^p \alpha_i P\delta_{a_i, \lambda_i} + v,$$

where $v \in H^2(\Omega) \cap H_0^1(\Omega) \cap T_w W_s(w)$ and satisfies

$$(V_0) : \langle v, \psi \rangle = 0 \text{ for } \psi \in \{P\delta_i, \frac{\partial P\delta_i}{\partial \lambda_i}, \frac{\partial P\delta_i}{\partial a_i}, i = 1, \dots, p\}.$$

Here, $P\delta_i = P\delta_{a_i, \lambda_i}$ and $\langle \cdot, \cdot \rangle$ denotes the inner product on $H^2(\Omega) \cap H_0^1(\Omega)$ defined by

$$\langle u, v \rangle = \int_{\Omega} \Delta u \Delta v.$$

The following Proposition deals with the v -part of u and shows that is negligible with respect to the concentration phenomenon.

Proposition 2.3 ([5], [4]) *There is a \mathcal{C}^1 -map which to each $(\alpha_i, a_i, \lambda_i)$ such that $\sum_{i=1}^p \alpha_i P\delta_{a_i, \lambda_i}$ belongs to $V(p, \varepsilon)$ associates $\bar{v} = \bar{v}(\alpha_i, a_i, \lambda_i)$ such that \bar{v} is the unique solution of the following minimization problem*

$$\min \left\{ J \left(\sum_{i=1}^p \alpha_i P\delta_{a_i, \lambda_i} + v \right), v \in H_0^1(\Omega) \text{ and satisfies } (V_0) \right\}.$$

Moreover, there exists a change of variables $v - \bar{v} \rightarrow V$ such that

$$J \left(\sum_{i=1}^p \alpha_i P\delta_{a_i, \lambda_i} + v \right) = J \left(\sum_{i=1}^p \alpha_i P\delta_{a_i, \lambda_i} + \bar{v} \right) + \|V\|^2.$$

We now state the definition of critical point at infinity.

Definition 2.4 [4] *A critical point at infinity of J is a limit of a non-compact flow line $u(s)$ of the gradient vector field $(-\partial J)$. By Propositions 2.1 and 2.2, $u(s)$ can be written as:*

$$u(s) = \sum_{i=1}^p \alpha_i(s) P\delta_{a_i(s), \lambda_i(s)} + v(s).$$

Denoting by $y_i = \lim_{s \rightarrow +\infty} a_i(s)$ and $\alpha_i = \lim_{s \rightarrow +\infty} \alpha_i(s)$, we then denote by

$$\sum_{i=1}^p \alpha_i P\delta_{y_i, \infty} \text{ or } (y_1, \dots, y_p)_\infty$$

such a critical point at infinity.

3 Expansion of the gradient of J

Let ρ be a positive small constant such that for any $y \in \mathcal{K}$, the expansion $(f)_\beta$ holds in $B(y, \rho)$. Let

$$\tilde{V}(p, \varepsilon) := \{u = \sum_{i=1}^p \alpha_i P\delta_{(a_i, \lambda_i)} \in V(p, \varepsilon), a_i \in B(y_{\ell_i}, \rho), y_{\ell_i} \in \mathcal{K}, \forall i = 1, \dots, p\}.$$

The following proposition gives the variation of J in $\tilde{V}(p, \varepsilon)$ with respect to $\lambda_i, i = 1, \dots, p$.

Proposition 3.1 *Assume that K satisfies $(f)_\beta$ condition for $\beta > 1$. For $u = \sum_{i=1}^p \alpha_i P\delta_{a_i, \lambda_i} \in \tilde{V}(p, \varepsilon)$, we have the following two estimates:*

$$\begin{aligned}
(a) \quad & \left\langle \partial J(u), \alpha_i \lambda_i \frac{\partial P\delta_{a_i, \lambda_i}}{\partial \lambda_i} \right\rangle = -2c_2 \frac{J(u)}{K(a_i)} \sum_{j \neq i} \alpha_i \alpha_j \left(\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{\frac{n-4}{2}}} \right) \\
& + O\left(\sum_{j=2}^{[\beta]} \frac{|a_i - y_{\ell_i}|^{\beta-j}}{\lambda_i^j} \right) + O\left(\frac{1}{\lambda_i^\beta} \right) + o\left(\sum_{j \neq i} \left(\varepsilon_{ij} + \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{\frac{n-4}{2}}} \right) \right). \\
(b) \quad & \left\langle \partial J(u), \alpha_i \lambda_i \frac{\partial P\delta_{a_i, \lambda_i}}{\partial \lambda_i} \right\rangle = -2c_2 \frac{J(u)}{K(a_i)} \sum_{j \neq i} \alpha_i \alpha_j \left(\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{\frac{n-4}{2}}} \right) \\
& + 2\alpha_i^2 \frac{J(u)}{K(a_i)} \begin{cases} \frac{n-4}{2} c_1 \frac{\sum_{k=1}^n b_k(y_{\ell_i})}{\lambda_i^{\beta(y_{\ell_i})}}, & \text{if } \beta(y_{\ell_i}) < n-4 \\ \frac{n-4}{2} c_1 \frac{\sum_{k=1}^n b_k(y_{\ell_i})}{\lambda_i^{\beta(y_{\ell_i})}} - c_2 \frac{H(y_{\ell_i}, y_{\ell_i})}{\lambda_i^{n-4}}, & \text{if } \beta(y_{\ell_i}) = n-4 \\ -c_2 \frac{H(y_{\ell_i}, y_{\ell_i})}{\lambda_i^{n-4}}, & \text{if } \beta(y_{\ell_i}) > n-4 \end{cases} \\
& + O(|a_i - y_{\ell_i}|^\beta) + o\left(\sum_{j \neq i} \left(\varepsilon_{ij} + \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{\frac{n-4}{2}}} \right) + \frac{1}{\lambda_i^{n-4}} \right).
\end{aligned}$$

Here $c_1 = \int_{\mathbb{R}^n} |z_1|^\beta \frac{|z|^2 - 1}{(1 + |z|^2)^{n+1}} dz$ and $c_2 = \int_{\mathbb{R}^n} \frac{|z|^2 - 1}{(1 + |z|^2)^n} dz$.

Proof. For $u = \sum_{i=1}^p \alpha_i P\delta_{a_i, \lambda_i} \in \tilde{V}(p, \varepsilon)$, we have:

$$\partial J(u) = 2J(u) \left(u + J(u) \Delta^{-1} (K u^{\frac{n+4}{n-4}}) \right).$$

Thus,

$$\begin{aligned}
\left\langle \partial J(u), \alpha_i \lambda_i \frac{\partial P\delta_{a_i, \lambda_i}}{\partial \lambda_i} \right\rangle &= 2J(u) \left[\sum_{j=1}^p \alpha_i \alpha_j \left\langle P\delta_{a_j, \lambda_j}, \lambda_i \frac{\partial P\delta_{a_i, \lambda_i}}{\partial \lambda_i} \right\rangle \right. \\
&\quad \left. - J(u)^{\frac{n}{n-4}} \int_{\Omega} K \left(\sum_{j=1}^p \alpha_j P\delta_{a_j, \lambda_j} \right)^{\frac{n+4}{n-4}} \alpha_i \lambda_i \frac{\partial P\delta_{a_i, \lambda_i}}{\partial \lambda_i} \right].
\end{aligned}$$

Using (2.3) and (2.4) and the fact that $J(u)^{\frac{n}{n-4}} \alpha_i^{\frac{4}{n-4}} K(a_i) = 1 + o(1)$, we get

$$\left\langle \partial J(u), \alpha_i \lambda_i \frac{\partial P\delta_{a_i, \lambda_i}}{\partial \lambda_i} \right\rangle = -2c_2 \frac{J(u)}{K(a_i)} \sum_{j \neq i} \alpha_i \alpha_j \left(\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{\frac{n-4}{2}}} \right)$$

$$-2\alpha_i^2 \frac{J(u)}{K(a_i)} \left(\int_{\Omega} K \delta_{a_j, \lambda_j}^{\frac{n+4}{n-4}} \lambda_i \frac{\partial \delta_{a_i, \lambda_i}}{\partial \lambda_i} + c_2 \frac{H(a_i, a_i)}{\lambda_i^{n-4}} \right) + o \left(\sum_{j \neq i} \left(\varepsilon_{ij} + \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{\frac{n-4}{2}}} \right) + \frac{1}{\lambda_i^{n-4}} \right).$$

Observe that

$$\delta_{a, \lambda}^{\frac{n+4}{n-4}} \lambda \frac{\partial \delta_{a, \lambda}}{\partial \lambda} = \frac{n-4}{2} \lambda^n \frac{1 - \lambda^2 |x - a|^2}{(1 + \lambda^2 |x - a|^2)^{n+1}}.$$

Let $\mu > 0$ such that $B(a_i, \mu) \subset B(y_{\ell_i}, \rho)$. We have

$$\int_{\Omega} K(x) \delta_{a_i, \lambda_i}^{\frac{n+4}{n-4}} \lambda_i \frac{\partial \delta_{a_i, \lambda_i}}{\partial \lambda_i} dx = \int_{B(a_i, \mu)} K(x) \delta_{a_i, \lambda_i}^{\frac{n+4}{n-4}} \lambda_i \frac{\partial \delta_{a_i, \lambda_i}}{\partial \lambda_i} dx + O\left(\frac{1}{\lambda_i^n}\right).$$

After a change of variables $z = \lambda_i(x - a_i)$,

$$\int_{B(a_i, \mu)} K(x) \delta_{a_i, \lambda_i}^{\frac{n+4}{n-4}} \lambda_i \frac{\partial \delta_{a_i, \lambda_i}}{\partial \lambda_i} dx = \frac{n-4}{2} \int_{B(0, \lambda_i \mu)} K(a_i + \frac{z}{\lambda_i}) \frac{1 - |z|^2}{(1 + |z|^2)^{n+1}} dz.$$

Using the following expansion of K around a_i ,

$$K(a_i + \frac{z}{\lambda_i}) = K(a_i) + \sum_{j=1}^{[\beta]} \frac{D^j K(a_i) (\frac{z}{\lambda_i})^j}{j!} + O\left(\left|\frac{z}{\lambda_i}\right|^\beta\right),$$

and the fact that $\int_{B(0, \lambda_i \mu)} K(a_i) \frac{1 - |z|^2}{(1 + |z|^2)^{n+1}} dz = O\left(\frac{1}{\lambda_i^n}\right)$, we get

$$\begin{aligned} \int_{B(a_i, \mu)} K(x) \delta_{a_i, \lambda_i}^{\frac{n+4}{n-4}} \lambda_i \frac{\partial \delta_{a_i, \lambda_i}}{\partial \lambda_i} dx &= \frac{n-4}{2} \sum_{j=1}^{[\beta]} \int_{B(0, \lambda_i \mu)} \frac{D^j K(a_i) (z)^j}{\lambda_i^j j!} \frac{1 - |z|^2}{(1 + |z|^2)^{n+1}} dz \\ &\quad + O\left(\frac{1}{\lambda_i^\beta} \int_{\mathbb{R}^n} |z_k|^\beta \frac{1 - |z|^2}{(1 + |z|^2)^{n+1}} dz\right) + O\left(\frac{1}{\lambda_i^n}\right). \end{aligned}$$

Observe that,

$$\int_{B(0, \lambda_i \mu)} DK(a_i)(z) \frac{1 - |z|^2}{(1 + |z|^2)^{n+1}} dz = 0.$$

Moreover, under $(f)_\beta$ -condition, we have

$$|D^j K(a_i)| = O(|a_i - y_{\ell_i}|^{\beta-j}).$$

Thus,

$$\int_{B(a_i, \mu)} K(x) \delta_{a_i, \lambda_i}^{\frac{n+4}{n-4}} \lambda_i \frac{\partial \delta_{a_i, \lambda_i}}{\partial \lambda_i} dx = O\left(\sum_{j=2}^{[\beta]} \frac{|a_i - y_{\ell_i}|^{\beta-j}}{\lambda_i^j}\right) + O\left(\frac{1}{\lambda_i^\beta}\right).$$

Hence, the estimate (a) of Proposition 3.1 follows.

For the estimate (b), $(f)_\beta$ -expansion yields

$$K(a_i + \frac{z}{\lambda_i}) = K(y_{\ell_i}) + \sum_{k=1}^n b_k \left| \frac{z_k}{\lambda_i} + (a_i - y_{\ell_i})_k \right|^\beta + o\left(\left| \frac{z}{\lambda_i} + (a_i - y_{\ell_i}) \right|^\beta\right)$$

Therefore,

$$\begin{aligned} \int_{B(a_i, \mu)} K(x) \delta_{a_i, \lambda_i}^{\frac{n+4}{n-4}} \lambda_i \frac{\partial \delta_{a_i, \lambda_i}}{\partial \lambda_i} dx &= \frac{n-4}{2} \frac{1}{\lambda_i^\beta} \sum_{k=1}^n b_k \int_{B(0, \lambda_i \mu)} \left| z_k + \lambda_i (a_i - y_{\ell_i})_k \right|^\beta \frac{1 - |z|^2}{(1 + |z|^2)^{n+1}} dz \\ &+ o\left(\frac{1}{\lambda_i^\beta} \int_{\mathbb{R}^n} |z_k|^\beta \frac{1 - |z|^2}{(1 + |z|^2)^{n+1}} dz\right) + o\left(|a_i - y_{\ell_i}|^\beta \int_{\mathbb{R}^n} |z_k|^\beta \frac{1 - |z|^2}{(1 + |z|^2)^{n+1}} dz\right). \\ &= \frac{n-4}{2} \frac{1}{\lambda_i^\beta} \sum_{k=1}^n b_k \int_{B(0, \lambda_i \mu)} |z_k|^\beta \frac{1 - |z|^2}{(1 + |z|^2)^{n+1}} dz + O(|a_i - y_{\ell_i}|^\beta) \\ &+ o\left(\frac{1}{\lambda_i^\beta} \int_{B(0, \lambda_i \mu)} |z_k|^\beta \frac{|1 - |z|^2|}{(1 + |z|^2)^{n+1}} dz\right) + o\left(\frac{1}{\lambda_i^{n-4}}\right). \end{aligned}$$

Observe that, for $\beta < n$,

$$\int_{B(0, \lambda_i \mu)} |z_k|^\beta \frac{1 - |z|^2}{(1 + |z|^2)^{n+1}} dz = \int_{\mathbb{R}^n} |z_k|^\beta \frac{1 - |z|^2}{(1 + |z|^2)^{n+1}} dz + O\left(\frac{1}{\lambda_i^{n-\beta}}\right) = -c_1 + O\left(\frac{1}{\lambda_i^{n-\beta}}\right).$$

For $\beta = n$,

$$\int_{B(0, \lambda_i \mu)} |z_k|^\beta \frac{1 - |z|^2}{(1 + |z|^2)^{n+1}} dz = O(\log \lambda_i).$$

Lastly, for $\beta > n$,

$$\int_{B(0, \lambda_i \mu)} |z_k|^\beta \frac{1 - |z|^2}{(1 + |z|^2)^{n+1}} dz = O\left(\frac{1}{\lambda_i^{n-\beta}}\right).$$

Therefore,

$$\int_{\Omega} K(x) \delta_{a_i, \lambda_i}^{\frac{n+4}{n-4}} \lambda_i \frac{\partial \delta_{a_i, \lambda_i}}{\partial \lambda_i} dx = \left(-\frac{n-4}{2} c_1 \frac{\sum_{k=1}^n b_k}{\lambda_i^\beta} + o\left(\frac{1}{\lambda_i^\beta}\right), \text{ if } \beta < n \right) + O(|a_i - y_{\ell_i}|^\beta) + o\left(\frac{1}{\lambda_i^{n-4}}\right). \quad (3.1)$$

This conclude the proof of Proposition 3.1. \square

Proposition 3.2 Assume that K satisfies $(f)_\beta$ condition for $\beta > 1$. Let $u = \sum_{i=1}^p \alpha_i P\delta_{a_i, \lambda_i} \in \tilde{V}(p, \varepsilon)$. For any $i = 1, \dots, p$ and $k = 1, \dots, n$, we have the following expansions.

$$(i) \quad \left\langle \partial J(u), \alpha_i \frac{1}{\lambda_i} \frac{\partial P\delta_{a_i, \lambda_i}}{\partial (a_i)_k} \right\rangle = -c_3 \alpha_i^2 J(u) \frac{b_k}{\lambda_i K(a_i)} \beta \operatorname{sign}(a_i - y_{\ell_i})_k |(a_i - y_{\ell_i})_k|^{\beta-1} \\ + O\left(\sum_{j=2}^{\min(n, \beta)} \frac{|a_i - y_{\ell_i}|^{\beta-j}}{\lambda_i^j}\right) + O\left(\frac{1}{\lambda_i^{\min(n, \beta)}}\right) + O\left(\frac{1}{\lambda_i^{n-1}}\right) + O\left(\sum_{j \neq i} \left|\frac{1}{\lambda_i} \frac{\partial \varepsilon_{ij}}{\partial a_i}\right|\right). \quad (3.2)$$

Moreover, if $\lambda_i |a_i - y_{\ell_i}|$ is bounded and $\beta < n + 1$, we have

$$(ii) \quad \left\langle \partial J(u), \alpha_i \frac{1}{\lambda_i} \frac{\partial P\delta_{a_i, \lambda_i}}{\partial (a_i)_k} \right\rangle = -(n-2) \alpha_i^2 J(u) \frac{b_k}{K(a_i) \lambda_i^\beta} \int_{\mathbb{R}^n} |z_k + \lambda_i (a_i - y_{\ell_i})_k|^\beta \\ \times \frac{z_k}{(1 + |z|^2)^{n+1}} dz + o\left(\frac{1}{\lambda_i^\beta}\right) + O\left(\sum_{j \neq i} \left|\frac{1}{\lambda_i} \frac{\partial \varepsilon_{ij}}{\partial a_i}\right|\right). \quad (3.3)$$

Here $c_3 = ((n-2)c_n^{\frac{2n}{n-4}}) \int_{\mathbb{R}^n} \frac{|z|^2}{(1 + |z|^2)^{n+1}} dz$.

Proof. We argue as in the proof of Proposition 3.1,

$$\left\langle \partial J(u), \alpha \frac{1}{\lambda} \frac{\partial P\delta_{a, \lambda}}{\partial a_k} \right\rangle = \frac{2\alpha^2 J(u)}{K(a)} \left(- \int_{\Omega} K(x) \delta_{a, \lambda}^{\frac{n+4}{n-4}} \frac{1}{\lambda} \frac{\partial \delta_{a, \lambda}}{\partial a_k} dx + O\left(\frac{\frac{\partial H}{\partial a_k}(a, a)}{\lambda^{n-1}}\right) + o\left(\frac{1}{\lambda^{n-1}}\right) \right) \\ + O\left(\sum_{j \neq i} \left|\frac{1}{\lambda_i} \frac{\partial \varepsilon_{ij}}{\partial a_i}\right|\right) \\ = -\frac{2\alpha^2 J(u)}{K(a)} \int_{B(a, \mu)} K(x) \delta_{a, \lambda}^{\frac{n+4}{n-4}} \frac{1}{\lambda} \frac{\partial \delta_{a, \lambda}}{\partial a_k} dx + O\left(\frac{1}{\lambda^{n-1}}\right) + O\left(\sum_{j \neq i} \left|\frac{1}{\lambda_i} \frac{\partial \varepsilon_{ij}}{\partial a_i}\right|\right).$$

Observe that

$$\delta_{a, \lambda}^{\frac{n+4}{n-4}} \frac{1}{\lambda} \frac{\partial \delta_{a, \lambda}}{\partial a_k} = (n-4) c_n^{\frac{2n}{n-4}} \frac{\lambda^{n+1} (x-a)_k}{(1 + \lambda^2 |x-a|^2)^{n+1}}.$$

A change of variables $z = \lambda(x-a)$ yields

$$\int_{B(a, \mu)} K(x) \delta_{a, \lambda}^{\frac{n+4}{n-4}} \frac{1}{\lambda} \frac{\partial \delta_{a, \lambda}}{\partial a_k} dx = (n-4) c_n^{\frac{2n}{n-4}} \int_{B(0, \lambda\mu)} K(a + z/\lambda) \frac{z_k}{(1 + |z|^2)^{n+1}} dz.$$

To get the first expansion of Proposition 3.2, we expand K as follows

$$K(a + z/\lambda) = K(a) + \sum_{j=1}^{\min(n,\beta)} \frac{D^j K(a)(z/\lambda)^j}{j!} + O\left(\frac{|z|^{\min(\beta,n)}}{\lambda^{\min(\beta,n)}}\right).$$

Using the fact that $\int_{B(0,\lambda\mu)} K(a) \frac{z_k}{(1+|z|^2)^{n+1}} dz = 0$, we get

$$\begin{aligned} \int_{B(a,\mu)} K(x) \delta_{a,\lambda}^{\frac{n+4}{n-4}} \frac{1}{\lambda} \frac{\partial \delta_{a,\lambda}}{\partial a_k} dx &= (n-4) c_n^{\frac{2n}{n-4}} \frac{1}{\lambda} \int_{B(0,\lambda\mu)} \frac{DK(a)(z) \cdot z_k}{(1+|z|^2)^{n+1}} dz \\ &+ O\left(\sum_{j=2}^{\min(n,\beta)} \frac{1}{\lambda^j} \int_{\mathbb{R}^n} \frac{|D^j K(a)| |z|^{j+1}}{(1+|z|^2)^{n+1}} dz\right) + O\left(\frac{1}{\lambda^{\min(n,\beta)}} \int_{\mathbb{R}^n} \frac{|z|^{\min(n,\beta)+1}}{(1+|z|^2)^{n+1}} dz\right). \end{aligned}$$

Observe that

$$\begin{aligned} \int_{B(0,\lambda\mu)} \frac{DK(a)(z) \cdot z_k}{(1+|z|^2)^{n+1}} dz &= \sum_{j=1}^n \frac{\partial K(a)}{\partial x_j} \int_{B(0,\lambda\mu)} \frac{z_j z_k}{(1+|z|^2)^{n+1}} dz \\ &= \frac{1}{n} \frac{\partial K(a)}{\partial x_k} \int_{B(0,\lambda\mu)} \frac{|z|^2}{(1+|z|^2)^{n+1}} dz = \frac{1}{n} \frac{\partial K(a)}{\partial x_k} \int_{\mathbb{R}^n} \frac{|z|^2}{(1+|z|^2)^{n+1}} dz + O\left(\frac{1}{\lambda^n}\right), \end{aligned}$$

since $\int_{B(0,\lambda\mu)} \frac{z_j z_k}{(1+|z|^2)^{n+1}} dz = 0, \forall j \neq k$. Using now the fact that $a \in B(y, \rho)$, we derive from $(f)_\beta$ -condition that

$$\begin{aligned} \frac{\partial K}{\partial x_k}(a) &= b_k \beta \operatorname{sign}(a-y)_k |(a-y)_k|^{\beta-1} + \frac{\partial R}{\partial x_k}(a-y) \\ &= b_k \beta \operatorname{sign}(a-y)_k |(a-y)_k|^{\beta-1} + o(|x-y|^{\beta-1}). \end{aligned}$$

Moreover, for every $j = 2, \dots, \tilde{\beta}$,

$$|D^j K(a)| = O(|a-y|^{\beta-j}).$$

Thus,

$$\begin{aligned} \int_{B(a,\mu)} K(x) \delta_{a,\lambda}^{\frac{n+4}{n-4}} \frac{1}{\lambda} \frac{\partial \delta_{a,\lambda}}{\partial a_k} dx &= c_3 \frac{b_k}{\lambda} \operatorname{sign}(a-y)_k |(a-y)_k|^{\beta-1} + o\left(\frac{|a-y|^{\beta-1}}{\lambda}\right) \\ &+ O\left(\sum_{j=2}^{\tilde{\beta}} \frac{|a-y|^{\beta-j}}{\lambda^j}\right) + O\left(\frac{1}{\lambda^{\min(n,\beta)}}\right). \end{aligned}$$

This finishes the proof of (a) of Proposition 3.2. Concerning the estimate (b), it follows from the above arguments and the following estimate

$$\int_{B(a,\mu)} K(x) \delta_{a,\lambda}^{\frac{n+4}{n-4}} \frac{1}{\lambda} \frac{\partial \delta_{a,\lambda}}{\partial a_k} dx = (n-4) c_n^{\frac{2n}{n-4}} \frac{b_k}{\lambda} \int_{\mathbb{R}^n} |z_k + \lambda(a-y)_k|^\beta \frac{z_k}{(1+|z|^2)^{n+1}} dz + o\left(\frac{1}{\lambda^\beta}\right).$$

This finishes the proof of Proposition 3.2. \square

4 Lack of compactness and critical points at infinity

In the first part of this section, we focus on $\tilde{V}(1, \varepsilon)$; the neighborhood of critical points at infinity consisting by single masses. We study the concentration phenomenon in this set and we identify the related critical points at infinity. Let $\rho > 0$ small enough such that for any $y \in \mathcal{K}$, the expansion $(f)_\beta$ holds in $B(y, \rho)$ and let:

$$\begin{aligned} \tilde{V}_1(1, \varepsilon) &= \{u = \alpha \delta_{(a,\lambda)} \in \tilde{V}(1, \varepsilon), a \in B(y, \rho), y \in \mathcal{K} \text{ with } \beta = \beta(y) < n-4\}, \\ \tilde{V}_2(1, \varepsilon) &= \{u = \alpha \delta_{(a,\lambda)} \in \tilde{V}(1, \varepsilon), a \in B(y, \rho), y \in \mathcal{K} \text{ with } \beta = \beta(y) = n-4\}, \\ \tilde{V}_3(1, \varepsilon) &= \{u = \alpha \delta_{(a,\lambda)} \in \tilde{V}(1, \varepsilon), a \in B(y, \rho), y \in \mathcal{K} \text{ with } \beta = \beta(y) > n-4\} \end{aligned}$$

As in [4], see also [1], the characterization of the critical points at infinity in $\tilde{V}(1, \varepsilon)$ is obtained through the construction of a suitable decreasing pseudo-gradient satisfying the P.S condition as long as the concentration point $a(s)$ does not enter in a neighborhood of $y \in \mathcal{K}_{<n-4}^+ \cup \mathcal{K}_{n-4}^+ \cup \mathcal{K}_{>n-4}$.

let δ be a small positive constant and let θ_1, θ_2 and θ_3 be the following three cut-off functions

$$\begin{aligned} \theta_1 : \mathbb{R} &\longrightarrow \mathbb{R} \\ t &\longmapsto \begin{cases} 1 & \text{if } |t| \leq \frac{\delta}{2} \\ 0 & \text{if } |t| \geq \delta. \end{cases} \end{aligned}$$

$$\begin{aligned} \theta_2 : \mathbb{R} &\longrightarrow \mathbb{R} \\ t &\longmapsto \begin{cases} 1 & \text{if } \frac{\delta}{2} \leq |t| \leq \frac{1}{\delta} \\ 0 & \text{if } |t| \in [0, \frac{\delta}{4}] \cup [\frac{2}{\delta}, +\infty[. \end{cases} \end{aligned}$$

$$\begin{aligned} \theta_3 : \mathbb{R} &\longrightarrow \mathbb{R} \\ t &\longmapsto \begin{cases} 1 & \text{if } |t| \geq \frac{1}{\delta} \\ 0 & \text{if } |t| \leq \frac{1}{2\delta}. \end{cases} \end{aligned}$$

• **Pseudo-gradient in $\tilde{V}_1(1, \varepsilon)$:**

Let W_1 be the following vector field. $\forall u = \alpha \delta_{(a, \lambda)} \in \tilde{V}_1(1, \varepsilon)$,

$$\begin{aligned} W_1(u) = & -\theta_1(\lambda|a-y|)\left(\sum_{k=1}^n b_k\right)\alpha\lambda\frac{\partial\delta_{(a,\lambda)}}{\partial\lambda} + \theta_3(\lambda|a-y|)\left(\sum_{k=1}^n b_k\right)\text{sign}(a-y)_k\alpha\frac{1}{\lambda}\frac{\partial\delta_{(a,\lambda)}}{\partial a_k} \\ & + \theta_2(\lambda|a-y|)\left(\sum_{k=1}^n b_k\right)\int_{\mathbb{R}^n}|z_k+\lambda(a-y)_k|^\beta\frac{z_k}{(1+|z|^2)^{n+1}}dz\alpha\frac{1}{\lambda}\frac{\partial\delta_{(a,\lambda)}}{\partial a_k}. \end{aligned}$$

We claim that

$$\langle \partial J(u), W_1(u) \rangle \leq -c\left(\frac{1}{\lambda^\beta} + \frac{|\nabla K(a)|}{\lambda}\right). \quad (4.1)$$

Indeed, if $\lambda|a-y| \leq \delta$, by Proposition 3.1, we have

$$\langle \partial J(u), -\left(\sum_{k=1}^n b_k\right)\alpha\lambda\frac{\partial\delta_{(a,\lambda)}}{\partial\lambda} \rangle \leq -c\frac{\left(\sum_{k=1}^n b_k\right)^2}{\lambda^\beta}, \quad (4.2)$$

since $|a-y|^\beta = o\left(\frac{1}{\lambda^\beta}\right)$ as δ small enough. Observe that under (f_β) -condition, we have

$$|\nabla K(a)| = O\left(|a-y|^{\beta-1}\right), \quad (4.3)$$

thus

$$\frac{|\nabla K(a)|}{\lambda} = O\left(\frac{1}{\lambda^\beta}\right), \quad \text{if } \lambda|a-y| \text{ is bounded.} \quad (4.4)$$

Therefore, we can appear $-\frac{|\nabla K(a)|}{\lambda}$ in the upper bound of (4.2) and we obtain

$$\langle \partial J(u), -\left(\sum_{k=1}^n b_k\right)\alpha\lambda\frac{\partial\delta_{(a,\lambda)}}{\partial\lambda} \rangle \leq -c\left(\frac{1}{\lambda^\beta} + \frac{|\nabla K(a)|}{\lambda}\right). \quad (4.5)$$

If $\lambda|a-y| \in [\frac{\delta}{4}, \frac{2}{\delta}]$, by the second expansion of Proposition 3.2, we obtain

$$\begin{aligned} & \langle \partial J(u), \sum_{k=1}^n b_k \int_{\mathbb{R}^n} |z_k + \lambda(a-y)_k|^\beta \frac{z_k}{(1+|z|^2)^{n+1}} dz \alpha \frac{1}{\lambda} \frac{\partial\delta_{(a,\lambda)}}{\partial a_k} \rangle \\ & \leq -\frac{c}{\lambda^\beta} \left(\int_{\mathbb{R}^n} |z_{k_a} + \lambda(a-y)_{k_a}|^\beta \frac{z_{k_a}}{(1+|z|^2)^{n+1}} dz \right)^2 + o\left(\frac{1}{\lambda^\beta}\right), \end{aligned}$$

where $|(a-y)_{k_a}| = \max_{1 \leq k \leq n} |(a-y)_k|$. Since $\lambda|(a-y)_{k_a}| \geq \frac{1}{\sqrt{n}}\frac{\delta}{4}$, we get

$$\left(\int_{\mathbb{R}^n} |z_{k_a} + \lambda(a-y)_{k_a}|^\beta \frac{z_{k_a}}{(1+|z|^2)^{n+1}} dz \right)^2 \geq c_\delta > 0.$$

Thus,

$$\begin{aligned} \langle \partial J(u), \sum_{k=1}^n b_k \int_{\mathbb{R}^n} |z_k + \lambda(a - y)_k|^\beta \frac{z_k}{(1 + |z|^2)^{n+1}} dz \alpha \frac{1}{\lambda} \frac{\partial \delta_{(a, \lambda)}}{\partial a_k} \rangle &\leq -\frac{c_1}{\lambda^\beta}, \\ &\leq -c \left(\frac{1}{\lambda^\beta} + \frac{|\nabla K(a)|}{\lambda} \right). \end{aligned}$$

Lastly, if $\lambda|a - y| \geq \frac{1}{2\delta}$, by the first expansion of Proposition 3.2, we have

$$\begin{aligned} \langle \partial J(u), \sum_{k=1}^n b_k \operatorname{sign}(a - y)_k \alpha \frac{1}{\lambda} \frac{\partial \delta_{(a, \lambda)}}{\partial a_k} \rangle &\leq -c \sum_{k=1}^n b_k^2 \frac{|a - y|^{\beta-1}}{\lambda} \\ &\quad + O\left(\sum_{j=2}^{\tilde{\beta}} \frac{|a - y|^{\beta-j}}{\lambda^j} \right) + O\left(\frac{1}{\lambda^\beta} \right). \end{aligned}$$

Observe that for every $j = 2, \dots, \tilde{\beta}$

$$\frac{|a - y|^{\beta-j}}{\lambda^j} = o\left(\frac{|a - y|^{\beta-1}}{\lambda} \right), \text{ as } \delta \text{ small enough.}$$

Also,

$$\frac{1}{\lambda^\beta} = o\left(\frac{|a - y|^{\beta-1}}{\lambda} \right), \text{ as } \delta \text{ small enough.} \quad (4.6)$$

Then, we obtain

$$\langle \partial J(u), \sum_{k=1}^n b_k \operatorname{sign}(a - y)_k \alpha \frac{1}{\lambda} \frac{\partial \delta_{(a, \lambda)}}{\partial a_k} \rangle \leq -c \frac{|a - y|^{\beta-1}}{\lambda},$$

since $\sum_{k=1}^n |(a - y)_k|^{\beta-1} \sim |a - y|^{\beta-1}$. Now by (4.3) and (4.6), we derive from the above inequality that

$$\langle \partial J(u), \sum_{k=1}^n b_k \operatorname{sign}(a - y)_k \alpha \frac{1}{\lambda} \frac{\partial \delta_{(a, \lambda)}}{\partial a_k} \rangle \leq -c \left(\frac{1}{\lambda^\beta} + \frac{|\nabla K(a)|}{\lambda} \right).$$

Hence claim (4.1) follows.

• **Pseudo-gradient in $\tilde{V}_2(1, \varepsilon)$:**

Let W_2 be the following vector field. $\forall u = \alpha \delta_{(a, \lambda)} \in \tilde{V}_2(1, \varepsilon)$,

$$W_2(u) = \theta_1(\lambda|a - y|) \alpha \lambda \frac{\partial \delta_{(a, \lambda)}}{\partial \lambda} + \theta_2(\lambda|a - y|) \sum_{k=1}^n b_k \int_{\mathbb{R}^n} \frac{z_k |z_k + \lambda(a - y)_k|^\beta}{(1 + |z|^2)^{n+1}} dz$$

$$+\theta_3(\lambda|a-y|) \sum_{k=1}^n b_k \operatorname{sign}(a-y)_k \alpha \frac{1}{\lambda} \frac{\partial \delta_{(a,\lambda)}}{\partial a_k}.$$

Observe that, if $\lambda|a-y| \leq \delta$, by the expansion of Proposition 3.1, we get

$$\begin{aligned} \langle \partial J(u), \alpha \lambda \frac{\partial \delta_{(a,\lambda)}}{\partial \lambda} \rangle &\leq -c \frac{H(a,a)}{\lambda^{n-4}} + O(|a-y|^{n-4}) \\ &\leq -\frac{c}{\lambda^{n-4}}, \end{aligned}$$

since $|a-y|^{n-4} = o\left(\frac{1}{\lambda^{n-4}}\right)$ as δ small enough and H is a positive regular function on Ω^2 . Thus by (4.4), we obtain

$$\langle \partial J(u), \alpha \lambda \frac{\partial \delta_{(a,\lambda)}}{\partial \lambda} \rangle \leq -c \left(\frac{1}{\lambda^{n-4}} + \frac{|\nabla K(a)|}{\lambda} \right).$$

If $\lambda|a-y| \geq \frac{\delta}{2}$, we proceed exactly as in $\tilde{V}_1(1, \varepsilon)$. We therefore obtain

$$\langle \partial J(u), W_2(u) \rangle \leq -c \left(\frac{1}{\lambda^{n-4}} + \frac{|\nabla K(a)|}{\lambda} \right). \quad (4.7)$$

• **Pseudo-gradient in $\tilde{V}_3(1, \varepsilon)$:**

Let W_3 be the following vector field. $\forall u = \alpha \delta_{(a,\lambda)} \in \tilde{V}_3(1, \varepsilon)$,

$$W_3(u) = \theta_1(\lambda^{n-4}|a-y|^\beta) \alpha \lambda \frac{\partial \delta_{(a,\lambda)}}{\partial \lambda} + (1-\theta_1)(\lambda^{n-4}|a-y|^\beta) \sum_{k=1}^n b_k \operatorname{sign}(a-y)_k \alpha \frac{1}{\lambda} \frac{\partial \delta_{(a,\lambda)}}{\partial a_k}.$$

We claim that

$$\langle \partial J(u), W_3(u) \rangle \leq -c \left(\frac{1}{\lambda^{\min(n-1, \beta)}} + \frac{|\nabla K(a)|}{\lambda} \right). \quad (4.8)$$

Indeed, if $\lambda^{n-4}|a-y|^\beta \leq \delta$ in the expansion of Proposition 3.1, we have

$$O(|a-y|^\beta) = o\left(\frac{1}{\lambda^{n-4}}\right), \quad \text{taking } \delta \text{ small enough,}$$

Thus,

$$\langle \partial J(u), \alpha \lambda \frac{\partial \delta_{(a,\lambda)}}{\partial \lambda} \rangle \leq -\frac{c}{\lambda^{n-4}}.$$

Moreover,

$$\frac{|\nabla K(a)|}{\lambda} = O\left(\frac{|a-y|^{\beta-1}}{\lambda}\right) = O\left(\frac{1}{\lambda^{1+\frac{(n-4)(\beta-1)}{\beta}}}\right) = o\left(\frac{1}{\lambda^{n-4}}\right)$$

Therefore, we can appear $\frac{\nabla K(a)}{\lambda}$ in the latest upper bound. Hence

$$\begin{aligned} < \partial J(u), \alpha \lambda \frac{\partial \delta_{(a,\lambda)}}{\partial \lambda} > \leq -c \left(\frac{1}{\lambda^{n-2}} + \frac{|\nabla K(a)|}{\lambda} \right) \\ & \leq -c \left(\frac{1}{\lambda^{\min(n-1,\beta)}} + \frac{|\nabla K(a)|}{\lambda} \right). \end{aligned}$$

Now, if $\lambda^{n-4}|a-y|^\beta > \frac{\delta}{2}$, by the first expansion of Proposition 3.2, we have

$$\begin{aligned} < \partial J(u), \sum_{k=1}^n b_k \operatorname{sign}(a-y)_k \alpha \frac{1}{\lambda} \frac{\partial \delta_{(a,\lambda)}}{\partial a_k} > \leq -c \sum_{k=1}^n b_k^2 \frac{|a-y|^{\beta-1}}{\lambda} + O \left(\sum_{j=2}^{\tilde{\beta}} \frac{|a-y|^{\beta-j}}{\lambda^j} \right) \\ & + O \left(\frac{1}{\lambda^{\min(\beta,n)}} \right) + O \left(\frac{1}{\lambda^{n-1}} \right). \end{aligned}$$

Observe that for every $j = 2, \dots, n$

$$\frac{|a-y|^{\beta-j}}{\lambda^j} = o \left(\frac{|a-y|^{\beta-1}}{\lambda} \right), \text{ as } \lambda \text{ goes to } +\infty.$$

Indeed,

$$\frac{|a-y|^{\beta-j}}{\lambda^j} \frac{\lambda}{|a-y|^{\beta-1}} = \frac{1}{(\lambda|a-y|)^{j-1}} \leq \left(\frac{2}{\delta} \right)^{\frac{j-1}{\beta}} \frac{1}{\lambda^{(j-1)(1-\frac{n-4}{\beta})}},$$

which goes to zero when λ goes to $+\infty$. In addition,

$$\frac{1}{\lambda^{n-1}} = o \left(\frac{|a-y|^{\beta-1}}{\lambda} \right), \text{ and } \frac{1}{\lambda^{\min(\beta,n)}} = o \left(\frac{|a-y|^{\beta-1}}{\lambda} \right) \text{ as } \lambda \rightarrow +\infty. \quad (4.9)$$

Therefore,

$$< \partial J(u), \sum_{k=1}^n b_k \operatorname{sign}(a-y)_k \alpha \frac{1}{\lambda} \frac{\partial \delta_{(a,\lambda)}}{\partial a_k} > \leq -c \frac{|a-y|^{\beta-1}}{\lambda}$$

and by (4.2) and (4.9), we obtain

$$< \partial J(u), \sum_{k=1}^n b_k \operatorname{sign}(a-y)_k \alpha \frac{1}{\lambda} \frac{\partial \delta_{(a,\lambda)}}{\partial a_k} > \leq -c \left(\frac{1}{\lambda^{\min(n-1,\beta)}} + \frac{|\nabla K(a)|}{\lambda} \right). \quad (4.10)$$

Hence our claim (4.8) follows.

Proof of Theorem 1.3. Let W the vector field in $\tilde{V}(1, \varepsilon)$ defined by convex combination of W_1, W_2 and W_3 . By (4.1), (4.7) and (4.8), we have

$$< \partial J(u), W(u) > \leq -c \left(\frac{1}{\lambda^{\min(n-1,\beta)}} + \frac{|\nabla K(a)|}{\lambda} \right).$$

In the above construction of W , we observe that the Palais-Smale condition is satisfied along the decreasing flow lines of the pseudo-gradient W as long as the concentration points $a(s)$ of the flow do not enter in some neighborhood of any critical point $y \in \mathcal{K}_{<n-4}^+ \cup \mathcal{K}_{n-4}^+ \cup \mathcal{K}_{>n-4}$, since $\lambda(s)$ decreases on the flow line in this region. However, if $a(s)$ is near a critical point $y \in \mathcal{K}_{<n-4}^+ \cup \mathcal{K}_{n-4}^+ \cup \mathcal{K}_{>n-4}$, $\lambda(s)$ increases on the flow line and goes to $+\infty$. Thus, we obtain a critical point at infinity.

In this statement, the functional J can be expended after a suitable change of variables as

$$J(\alpha P\delta_{a,\lambda} + \bar{v}) = J(\tilde{\alpha} P\delta_{\tilde{a},\tilde{\lambda}}) = \frac{S_n}{\tilde{\alpha}^{\frac{8}{n-4}} (K(\tilde{a}))^{\frac{n-4}{4}}} \left(1 + \frac{1}{\tilde{\lambda}^\beta}\right).$$

Thus, the index of such critical point at infinity is $n - \tilde{i}(y)$. Since J behaves in this region as $\frac{1}{K^{\frac{n-4}{4}}}$. This conclude the proof of Theorem 1.3. \square

In the second part of this section, we focus on $\tilde{V}(p, \varepsilon), p \geq 2$. We characterize the critical points at infinity in these sets in order to give a complete description of the loss of compactness of problem (1.1) under $(f)_\beta$ -condition, where $\beta \in (1, n-4]$. Let

$$\begin{aligned} \tilde{V}_{<n-4}(p, \varepsilon) &= \left\{ u = \sum_{i=1}^p \alpha_i \delta_{(a_i, \lambda_i)} + v \in \tilde{V}(p, \varepsilon), \ a_i \in B(y_{\ell_i}, \rho), \ y_{\ell_i} \in \mathcal{K} \right. \\ &\quad \left. \text{with } \beta = \beta(y_{\ell_i}) < n-4, \forall i = 1, \dots, p \right\}, \\ \tilde{V}_{n-4}(p, \varepsilon) &= \left\{ u = \sum_{i=1}^p \alpha_i \delta_{(a_i, \lambda_i)} + v \in \tilde{V}(p, \varepsilon), \ a_i \in B(y_{\ell_i}, \rho), \ y_{\ell_i} \in \mathcal{K} \right. \\ &\quad \left. \text{with } \beta = \beta(y_{\ell_i}) = n-4, \forall i = 1, \dots, p \right\}. \end{aligned}$$

We introduce now the following two Lemmas.

Lemma 4.1 *There exists a pseudo-gradient W_1 in $\tilde{V}_{<n-4}(p, \varepsilon)$ such that for any $u = \sum_{i=1}^p \alpha_i P\delta_{a_i, \lambda_i} \in \tilde{V}_{<n-4}(p, \varepsilon)$, we have*

$$\langle \partial J(u), W_1(u) \rangle \leq -c \left(\sum_{i=1}^p \left(\frac{1}{\lambda_i^\beta} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{j \neq i} \varepsilon_{ij} \right).$$

Moreover, the only situation when the $\lambda_i(s), i = 1, \dots, p, s \geq 0$, are not bounded is when $a_i(s)$ goes to $y_{\ell_i} \in \mathcal{K}_{<n-4}^+, \forall i = 1, \dots, p$ with $y_{\ell_i} \neq y_{\ell_j}, \forall i \neq j$.

Lemma 4.2 *There exists a pseudo-gradient W_2 in $\tilde{V}_{n-4}(p, \varepsilon)$ such that for any $u = \sum_{i=1}^p \alpha_i P\delta_{a_i, \lambda_i} \in \tilde{V}_{n-4}(p, \varepsilon)$, we have*

$$\langle \partial J(u), W_2(u) \rangle \leq -c \left(\sum_{i=1}^p \left(\frac{1}{\lambda_i^{n-4}} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{j \neq i} \varepsilon_{ij} \right).$$

Moreover, the only situation when the $\lambda_i(s), i = 1, \dots, p, s \geq 0$, are not bounded is when $a_i(s)$ goes to $y_{\ell_i} \in \mathcal{K}_{n-4}^+, \forall i = 1, \dots, p$ with $y_{\ell_i} \neq y_{\ell_j}, \forall i \neq j$ and $\rho(y_{\ell_1}, \dots, y_{\ell_p}) > 0$.

The proof of Lemmas 4.1 and 4.2 will be given at the end of this section. We now state the proof of Theorem 1.3.

Proof of Theorem 1.3. It follows from the following Lemma.

Lemma 4.3 *Under the assumption that K satisfies (B) and $(f)_\beta$ for $\beta \in (1, n-4]$, there exists a pseudo-gradient W in $\tilde{V}(p, \varepsilon)$ such that for any $u = \sum_{i=1}^p \alpha_i P\delta_{a_i, \lambda_i} \in \tilde{V}(p, \varepsilon)$, we have*

$$\langle \partial J(u), \tilde{W}(u) \rangle \leq -c \left(\sum_{i=1}^p \left(\frac{1}{\lambda_i^\beta} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{j \neq i} \varepsilon_{ij} \right). \quad (4.11)$$

Moreover, the only case where $\lambda_i(s), i = 1, \dots, p, s \geq 0$ are not bounded is when $a_i(s)$ goes to $y_{\ell_i} \in \mathcal{K}_{<n-4}^+ \cup \mathcal{K}_{n-4}^+, \forall i = 1, \dots, p$ with $y_{\ell_i} \neq y_{\ell_j}, \forall i \neq j$ and $(y_{\ell_1}, \dots, y_{\ell_p}) \in \mathcal{C}_{n-4}^\infty \cup \mathcal{C}_{<n-4}^\infty \cup (\mathcal{C}_{n-4}^\infty \times \mathcal{C}_{<n-4}^\infty)$.

Proof of Lemma 4.3. Let $u = \sum_{i=1}^p \alpha_i P\delta_{a_i, \lambda_i} \in \tilde{V}(p, \varepsilon), p \geq 2$. By Lemmas 4.1 and 4.2, it remains only to consider the case where

$$u = \sum_{i \in I_1} \alpha_i P\delta_{a_i, \lambda_i} + \sum_{i \in I_2} \alpha_i P\delta_{a_i, \lambda_i},$$

with $I_1 \neq \emptyset, I_2 \neq \emptyset, u_1 := \sum_{i \in I_1} \alpha_i P\delta_{a_i, \lambda_i} \in \tilde{V}_{<n-4}(\#I_1, \varepsilon)$ and $u_2 := \sum_{i \in I_2} \alpha_i P\delta_{a_i, \lambda_i} \in \tilde{V}_{n-4}(\#I_2, \varepsilon)$. We order the λ_i 's, we can assume that

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_p.$$

Three cases may occur.

- First case: $u_1 \notin \left\{ u = \sum_{j=1}^{\#I_1} \alpha_j P\delta_{a_j, \lambda_j} \in \tilde{V}_{<n-4}(\#I_1, \varepsilon) \text{ with } y_{\ell_j} \in \mathcal{K}_{<n-4}^+, y_{\ell_j} \neq y_{\ell_k}, \forall 1 \leq j \neq k \leq \#I_1 \right\}$.

Let $\widetilde{W}_1(u) = W_1(u_1)$ where W_1 is the pseudo-gradient defined in Lemma 4.1. Observe that the maximum of the $\lambda_i(s)$, $i \in I_1$ does not increase through W_1 . Moreover, by Lemma 4.1, we have

$$\begin{aligned} \langle \partial J(u), \widetilde{W}_1(u) \rangle \leq & -c \left(\sum_{i \in I_1} \left(\frac{1}{\lambda_i^\beta} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{i,j \in I_1, j \neq i} \varepsilon_{ij} \right) \\ & + O \left(\sum_{i \in I_1, j \notin I_1} \varepsilon_{ij} \right). \end{aligned} \quad (4.12)$$

Let k_0 be an index in I_1 such that

$$\lambda_{k_0}^{\beta(y_{\ell_{k_0}})} = \min \{ \lambda_i^{\beta(y_{\ell_i})}, i \in I_1 \}.$$

Define

$$\widetilde{I}_1 = \{ i, 1 \leq i \leq p, \text{ s.t. } \lambda_i^{\beta(y_{\ell_i})} \geq \frac{1}{10} \lambda_{k_0}^{\beta(y_{\ell_{k_0}})} \}.$$

Observe that $I_1 \subset \widetilde{I}_1$. Our first goal is to make appears in the upper bound of (4.12) all indices $i \in \widetilde{I}_1$. For each index i we define the following vector field.

$$X_i(u) = \sum_{k=1}^n b_k \operatorname{sign}(a_i - y_{\ell_i})_k \frac{1}{\lambda_i} \frac{\partial P \delta_{a_i, \lambda_i}}{\partial (a_i)_k}. \quad (4.13)$$

Using the first expansion of Lemma 3.2, we have

$$\begin{aligned} \langle \partial J(u), X_i(u) \rangle \leq & -\frac{c_3}{K(a_i)} \alpha_i^2 J(u) \sum_{k=1}^n b_k^2 \frac{|a_i - y_{\ell_i}|^{\beta-1}}{\lambda_i} \\ & + O \left(\sum_{j=2}^{[\beta]} \frac{|a_i - y_{\ell_i}|^{\beta-j}}{\lambda_i^j} \right) + O \left(\frac{1}{\lambda_i^\beta} \right) + o \left(\sum_{j \neq i} \varepsilon_{ij} \right), \end{aligned} \quad (4.14)$$

since $\left| \frac{1}{\lambda_i} \frac{\partial \varepsilon_{ij}}{\partial a_i} \right| = o(\varepsilon_{ij})$ if $|a_i - a_j| > \rho$. Let M be a large positive constant. Observe that if $|\lambda_i(a_i - y_{\ell_i})| \leq M$,

$$\frac{|a_i - y_{\ell_i}|^{\beta-j}}{\lambda_i^j} = O \left(\frac{1}{\lambda_i^\beta} \right),$$

and if $|\lambda_i(a_i - y_{\ell_i})| \geq M$,

$$\frac{|a_i - y_{\ell_i}|^{\beta-j}}{\lambda_i^j} = o \left(\frac{|a_i - y_{\ell_i}|^{\beta-1}}{\lambda_i} \right),$$

and

$$\frac{1}{\lambda_i^\beta} = o \left(\frac{|a_i - y_{\ell_i}|^{\beta-1}}{\lambda_i} \right) \text{ as } M \text{ large.}$$

Therefore, for $m_1 > 0$ very small, we get from (4.12) and (4.14)

$$\begin{aligned} \langle \partial J(u), \widetilde{W}_1(u) + m_1 \sum_{i \in \widetilde{I}_1 \setminus I_1} X_i(u) \rangle &\leq -c \left(\sum_{i \in \widetilde{I}_1} \left(\frac{1}{\lambda_i^\beta} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{i,j \in I_1, j \neq i} \varepsilon_{ij} \right) \\ &\quad + o \left(\sum_{j=1}^p \frac{1}{\lambda_j^\beta} \right), \end{aligned} \quad (4.15)$$

since under $(f)_\beta$ -condition $|\nabla K(a_i)| = O(|a_i - y_{\ell_i}|^{\beta-1})$ and

$$\varepsilon_{ij} = O \left(\frac{1}{(\lambda_i \lambda_j)^{\frac{n-2}{2}}} \right) = o \left(\frac{1}{\lambda_i^\beta} + \frac{1}{\lambda_j^\beta} \right), \forall i \in I_1, j \notin I_1.$$

In order to appear $-\sum_{i,j \in \widetilde{I}_1, j \neq i} \varepsilon_{ij}$ we will decrease all the $\lambda_i, i \in \widetilde{I}_1 \setminus I_1$ with different speed.

Let $Z_1(u) = \sum_{i \in \widetilde{I}_1 \setminus I_1} -2^i \lambda_i \frac{\partial P \delta_{a_i, \lambda_i}}{\partial \lambda_i}$. Observe that

$$2^i \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + 2^j \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} \leq -c \varepsilon_{ij}, \forall i < j.$$

Thus, using the first expansion of Proposition 3.1, we get

$$\langle \partial J(u), Z_1(u) \rangle \leq -c \sum_{i,j \in \widetilde{I}_1 \setminus I_1, j \neq i} \varepsilon_{ij} + O \left(\sum_{i \in \widetilde{I}_1 \setminus I_1} \frac{1}{\lambda_i^\beta} + \sum_{j=2}^{[\beta]} \frac{|a_i - y_{\ell_i}|^{\beta-j}}{\lambda_i^j} \right).$$

Therefore, for $m_2 > 2$ very small, we obtain

$$\begin{aligned} \langle \partial J(u), \widetilde{W}_1(u) + m_2 Z_1(u) + m_1 \sum_{i \in \widetilde{I}_1 \setminus I_1} X_i(u) \rangle &\leq -c \left(\sum_{i \in \widetilde{I}_1} \left(\frac{1}{\lambda_i^\beta} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{i,j \in \widetilde{I}_1, j \neq i} \varepsilon_{ij} \right) \\ &\quad + o \left(\sum_{j=1}^p \frac{1}{\lambda_j^\beta} \right). \end{aligned} \quad (4.16)$$

Now let R be the set of the remainder indices. So $R = \{i \in I_2, \lambda_i^\beta < \frac{1}{10} \lambda_{k_0}^\beta\}$ and denote $\widetilde{u} = \sum_{i \in R} \alpha_i P \delta_{a_i, \lambda_i}$. Observe that $\widetilde{u} \in \widetilde{V}_{n-2}(\sharp R, \varepsilon)$, therefore, we can apply the associated

vector field $W_2(\widetilde{u})$ defined in Lemma 4.2. For $\widetilde{W}_2(u) = W_2(\widetilde{u})$ we get by Lemma 4.2

$$\langle \partial J(u), \widetilde{W}_2(u) \rangle \leq -c \left(\sum_{i \in R} \left(\frac{1}{\lambda_i^\beta} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{i,j \in R, j \neq i} \varepsilon_{ij} \right)$$

$$+O\left(\sum_{i \in R, j \notin R} \varepsilon_{ij}\right).$$

We let in this case $W = \widetilde{W}_1 + \widetilde{W}_2 + m_2 Z_1 + m_1 \sum_{i \in \widetilde{I}_1 \setminus I_1} X_i(u)$. It satisfies

$$\langle \partial J(u), W(u) \rangle \leq -c \left(\sum_{i=1}^p \left(\frac{1}{\lambda_i^\beta} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{j \neq i} \varepsilon_{ij} \right). \quad (4.17)$$

- Second case: $u_2 \notin \left\{ u = \sum_{j=1}^{\#I_2} \alpha_j P \delta_{a_j, \lambda_j} \in \widetilde{V}_{n-4}(\#I_2, \varepsilon) \text{ with } y_{\ell_j} \in \mathcal{K}_{n-4}^+, y_{\ell_j} \neq y_{\ell_k}, \forall 1 \leq j \neq k \leq \#I_2 \text{ and } \rho(y_{\ell_1}, \dots, y_{\ell_p}) > 0 \right\}$.

Let $\widetilde{W}_2(u) = W_2(u_2)$ where W_2 is defined in Lemma 4.2. We then have:

$$\begin{aligned} \langle \partial J(u), \widetilde{W}_2(u) \rangle &\leq -c \left(\sum_{i \in I_2} \left(\frac{1}{\lambda_i^\beta} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{i, j \in I_2, j \neq i} \varepsilon_{ij} \right) \\ &\quad + O\left(\sum_{i \in I_2, j \notin I_2} \varepsilon_{ij}\right). \end{aligned}$$

As in the first case, we denote by k_0 the index of I_2 satisfying

$$\lambda_{k_0} = \min\{\lambda_i, i \in I_2\}$$

and we define

$$\widetilde{I}_2 = \{i, 1 \leq i \leq p, \text{ s.t. } \lambda_i^{\beta(y_{\ell_i})} \geq \frac{1}{10} \lambda_{k_0}^{n-4}\}$$

and $E = \widetilde{I}_2^c$. Let

$$Z_2(u) = \sum_{i \in \widetilde{I}_2 \setminus I_2} -2^i \lambda_i \frac{\partial P \delta_{a_i, \lambda_i}}{\partial \lambda_i},$$

$$\widetilde{W}_1(u) = W_1\left(\sum_{i \in E} \alpha_i P \delta_{a_i, \lambda_i}\right),$$

and $X_i(u)$ the vector field defined in (4.13). By the same computation of the first section, we get for $W = \widetilde{W}_2 + \widetilde{W}_1 + m_2 Z_2 + m_1 \sum_{i \in \widetilde{I}_2 \setminus I_2} X_i(u)$,

$$\langle \partial J(u), W(u) \rangle \leq -c \left(\sum_{i=1}^p \left(\frac{1}{\lambda_i^\beta} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{j \neq i} \varepsilon_{ij} \right).$$

- Third case: $u_1 \in \left\{ u = \sum_{j=1}^{\#I_1} \alpha_j P\delta_{a_j, \lambda_j} \in \tilde{V}_{<n-4}(\#I_1, \varepsilon) \text{ with } y_{\ell_j} \in \mathcal{K}_{<n-4}^+ \text{ and } y_{\ell_j} \neq y_{\ell_k}, \forall 1 \leq j \neq k \leq \#I_2 \right\}$ and $u_2 \in \left\{ u = \sum_{j=1}^{\#I_2} \alpha_j P\delta_{a_j, \lambda_j} \in \tilde{V}_{n-4}(\#I_2, \varepsilon) \text{ with } y_{\ell_j} \in \mathcal{K}_{n-4}^+, y_{\ell_j} \neq y_{\ell_k}, \forall 1 \leq j \neq k \leq \#I_2 \text{ and } \rho(y_{\ell_1}, \dots, y_{\ell_p}) > 0 \right\}$.

Let $\widetilde{W}_1(u) = W_1(u_1)$ and $\widetilde{W}_2(u) = W_2(u_2)$ where W_1 and W_2 are the vector fields defined in Lemmas 4.1 and 4.2 respectively. Using the above estimates, we get for $W = \widetilde{W}_1 + \widetilde{W}_2$

$$\langle \partial J(u), W(u) \rangle \leq -c \left(\sum_{i=1}^p \left(\frac{1}{\lambda_i^\beta} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{j \neq i} \varepsilon_{ij} \right).$$

This finishes the proof of Lemma 4.3 and then the proof of Theorem 1.3 follows. \square

\square

Proof of Lemma 4.1. We divide $\tilde{V}_1(p, \varepsilon)$ as follows. Let $\delta > 0$ and small.

$$W_1(p, \varepsilon) := \left\{ u = \sum_{j=1}^p \alpha_j P\delta_{a_j, \lambda_j} \in \tilde{V}_1(p, \varepsilon), y_{l_j} \neq y_{l_k}, \forall j \neq k, -\sum_{k=1}^n b_k(y_{l_j}) > 0, \text{ and } \lambda_j |a_j - y_{l_j}| < \delta, \forall j = 1, \dots, p \right\}.$$

$$W_2(p, \varepsilon) := \left\{ u = \sum_{j=1}^p \alpha_j P\delta_{a_j, \lambda_j} \in \tilde{V}_1(p, \varepsilon), y_{l_j} \neq y_{l_k}, \forall j \neq k, \lambda_j |a_j - y_{l_j}| < \delta, \forall j = 1, \dots, p \text{ and there exist at least } j_1 \text{ such that } -\sum_{k=1}^n b_k(y_{l_{j_1}}) < 0 \right\}.$$

$$W_3(p, \varepsilon) := \left\{ u = \sum_{j=1}^p \alpha_j P\delta_{a_j, \lambda_j} \in \tilde{V}_1(p, \varepsilon), y_{l_j} \neq y_{l_k}, \forall j \neq k, \text{ and there exist at least } j_1, s, t, \lambda_{j_1} |a_{j_1} - y_{l_{j_1}}| \geq \frac{\delta}{2} \right\}.$$

$$W_4(p, \varepsilon) := \left\{ u = \sum_{j=1}^p \alpha_j P\delta_{a_j, \lambda_j} \in \tilde{V}_1(p, \varepsilon), \text{ such that there exist } j \neq k \text{ with } y_{l_j} = y_{l_k} \right\}.$$

- Pseudo-gradient in $W_1(p, \varepsilon)$. In this region, we have $|a_j - y_{l_j}|^\beta = o\left(\frac{1}{\lambda_j^\beta}\right), \forall j = 1, \dots, p$

and $\frac{1}{(\lambda_j \lambda_k)^{\frac{n-2}{2}}} = o\left(\frac{1}{\lambda_j^\beta}\right) + o\left(\frac{1}{\lambda_k^\beta}\right)$ since $\beta < n - 2$. Thus, for $V_1(u) = \sum_{j=1}^p \lambda_j \frac{\partial P\delta_{a_j, \lambda_j}}{\partial \lambda_j}$, we have:

$$\langle \partial J(u), V_1(u) \rangle \leq -c \left(\sum_{i=1}^p \left(\frac{1}{\lambda_i^\beta} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{j \neq i} \varepsilon_{ij} \right),$$

since $\frac{|\nabla K(a_i)|}{\lambda_i} = o\left(\frac{1}{\lambda_j^\beta}\right)$.

• Pseudo-gradient in $W_2(p, \varepsilon)$. Let k_0 an index such that $\lambda_{k_0}^\beta = \min\{\lambda_j^\beta, \text{ such that } -\sum_{k=1}^n b_k(y_{l_j}) < 0\}$. We denote by $I_1 = \{j = 1, \dots, p, \text{ such that, } \lambda_j^\beta \leq \frac{1}{2}\lambda_{k_0}^\beta\}$. Observe that $\tilde{u} := \sum_{j \in I_1} \alpha_j P\delta_{a_j, \lambda_j} \in W_1(\#I_1, \varepsilon)$. Using the same previous technics, we get for $\tilde{V}_1(u) = V_1(\tilde{u})$,

$$\langle \partial J(u), \tilde{V}_1(u) - \sum_{j \notin I_1} \lambda_j \frac{\partial P\delta_{a_j, \lambda_j}}{\partial \lambda_j} \rangle \leq -c \left(\sum_{i=1}^p \left(\frac{1}{\lambda_i^\beta} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{j \neq i} \varepsilon_{ij} \right).$$

• Pseudo-gradient in $W_3(p, \varepsilon)$. Let k_1 an index such that $\lambda_{k_1}^\beta = \min\{\lambda_j^\beta, \text{ such that } \lambda_j |a_j - y_{l_j}| \geq \frac{\delta}{2}\}$ and let $J_1 = \{j = 1, \dots, p \text{ such that } \lambda_j^\beta \leq \frac{1}{2}\lambda_{k_1}^\beta\}$. For any $j \in J_1$, we have $\lambda_j |a_j - y_{l_j}| \leq \frac{\delta}{2}$. Let $\tilde{u} := \sum_{j \in J_1} \alpha_j P\delta_{a_j, \lambda_j}$, $\tilde{u} \in W_1(\#J_1, \varepsilon) \cup W_2(\#J_1, \varepsilon)$. We denote by $\tilde{W}_3(u) = V(\tilde{u})$ where V is the associated vector field to the above two regions. Using the second expansion of Lemma 3.2 and the previous technics, we get

$$\langle \partial J(u), \tilde{W}_3(u) + \sum_{i \notin J_1} \left(X_i(u) - \lambda_i \frac{\partial P\delta_{a_i, \lambda_i}}{\partial \lambda_i} \right) \rangle \leq -c \left(\sum_{i=1}^p \left(\frac{1}{\lambda_i^\beta} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{j \neq i} \varepsilon_{ij} \right).$$

Here $X_i(u) = \alpha_i \sum_{k=1}^n \frac{b_k}{\lambda_i} \int_{\mathbb{R}^n} \frac{|x_k + \lambda_i(a_i - y_{\ell_i})_k|^\beta}{(1 + \lambda_i|(a_i - y_{\ell_i})_k|^{\beta-1}) (1 + |x|^2)^{n+1}} dx$ if $\lambda_i|(a_i - y_{\ell_i})_k| \in [\frac{\delta}{2}, \frac{2}{\delta}]$ and $X_i(u)$ is defined by (4.13) otherwise.

• Pseudo-gradient in $W_4(p, \varepsilon)$. For any critical point y_ℓ of K , we denote $B_\rho = \{j = 1, \dots, p \text{ such that } a_j \in B(y_\ell, \rho)\}$. In this region, there exists at least ℓ such that $\#B_\ell \geq 2$. Let

$$J_1 = \{\ell, 1 \leq \ell \leq \#\mathcal{K}, \text{ such that } \#B_\ell \geq 2\}.$$

For any $\ell \in J_1$, we decrease all λ_j 's, $j \in B_\ell$ as follows. Let

$$\begin{aligned} \psi : \mathbb{R} &\longrightarrow \mathbb{R} \\ t &\longmapsto \begin{cases} 1 & \text{if } |t| \geq 1 \\ 0 & \text{if } |t| \leq \gamma. \end{cases} \end{aligned}$$

where $\gamma > 0$ very small. Define $\bar{\psi}(\lambda_j) = \sum_{i \neq j} \psi\left(\frac{\lambda_j}{\lambda_i}\right)$ for $j \in B_k$ and $k \in J_1$. By the first expansion of Lemma 3.1, we get

$$\langle \partial J(u), -\sum_{\ell \in J_1} \sum_{j \in B_\ell} \bar{\psi}(\lambda_j) \lambda_j \frac{\partial P \delta_{a_j, \lambda_j}}{\partial \lambda_j} \rangle \leq -c \sum_{\ell \in J_1} \sum_{i \neq j \in B_\ell} \varepsilon_{ij} + O\left(\sum_{i=1}^p \frac{1}{\lambda_i^\beta}\right).$$

To obtain the required upper bound, we set

$$I_1 := \{j; |\lambda_j| |a_i - y_{\ell_j}| \geq \delta\}.$$

If $I_1 \neq \emptyset$, we use the above vector field (defined in $W_3(p, \varepsilon)$) and using the expansions of Lemma 3.2, we obtain

$$\langle \partial J(u), -\sum_{\ell \in J_1} \sum_{j \in B_\ell} \bar{\psi}(\lambda_j) \lambda_j \frac{\partial P \delta_{a_j, \lambda_j}}{\partial \lambda_j} + \sum_{i \in I_1} X_i(u) \rangle \leq -c \left(\sum_{i=1}^p \left(\frac{1}{\lambda_i^\beta} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{j \neq i} \varepsilon_{ij} \right).$$

If $I_1 = \emptyset$, we denote by I_2 the set of indices constructed by 1 and all j such that $\lambda_j \sim \lambda_1$, (of the same order). We write $u = \sum_{i \in I_2} \alpha_i P \delta_{a_i, \lambda_i} + \sum_{i \notin I_2} \alpha_i P \delta_{a_i, \lambda_i}$. Observe that $\tilde{u} := \sum_{i \in I_2} \alpha_i P \delta_{a_i, \lambda_i} \in W_k(\#I_2, \varepsilon)$, $k = 1, 2$ or 3 . We then apply the associated vector field denoted $\widetilde{W}_4(u)$. We obtain

$$\begin{aligned} \langle \partial J(u), \widetilde{W}_4(u) \rangle &\leq -c \left(\sum_{i \in I_2} \left(\frac{1}{\lambda_i^\beta} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{i, j \in I_2, j \neq i} \varepsilon_{ij} \right) \\ &\quad + O\left(\sum_{i \in I_2, j \notin I_2} \varepsilon_{ij} \right) \end{aligned}$$

and therefore,

$$\langle \partial J(u), \widetilde{W}_4(u) - \sum_{\ell \in J_1} \sum_{j \in B_\ell} \bar{\psi}(\lambda_j) \lambda_j \frac{\partial P \delta_{a_j, \lambda_j}}{\partial \lambda_j} \rangle \leq -c \left(\sum_{i=1}^p \left(\frac{1}{\lambda_i^\beta} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{j \neq i} \varepsilon_{ij} \right).$$

This finishes the proof of Lemma 4.1. □

Proof of Lemma 4.2. The situation here is exactly the one of ([12], Proposition 3.7), so we omit the proof here. □

□

5 Proof of the existence results

5.1 Proof of Theorem 1.2

Using the result of Theorem 1.1, the critical points at infinity of the associated variational problem are in one to one correspondence with the elements $(y_1, \dots, y_p)_\infty, (y_1, \dots, y_p)_\infty \in \mathcal{C}^\infty$. For each $(y_1, \dots, y_p)_\infty \in \mathcal{C}^\infty$, we denote by $W_u^\infty(y_1, \dots, y_p)_\infty$; the unstable manifold of the critical points at infinity $(y_1, \dots, y_p)_\infty$. Recall that $i(y_1, \dots, y_p)_\infty$ the index of $(y_1, \dots, y_p)_\infty$ is equal to the dimension of $W_u^\infty(y_1, \dots, y_p)_\infty$. Using now the gradient flow of $(-\partial J)$ to deform Σ^+ . It follows then by deformation Lemma (see [?]), that

$$\Sigma^+ \simeq \bigcup_{(y_1, \dots, y_p)_\infty \in \mathcal{C}^\infty} W_u^\infty(y_1, \dots, y_p)_\infty \cup \bigcup_{w, \partial J(w)=0} W_u(w), \quad (5.1)$$

where \simeq denotes retracts by deformation. It follows from the above deformation retract that the problem (1.1) has necessary a solution w . Otherwise, it follows from (5.2) that

$$1 = \chi(\Sigma^+) = \sum_{(y_1, \dots, y_p)_\infty \in \mathcal{C}^\infty} (-1)^{i(y_1, \dots, y_p)_\infty},$$

where χ denotes the Euler-Poincare Characteristic. Such an equality contradicts the assumption of Theorem 1.2

5.2 Proof of Theorem 1.4

Let

$$J_1(u) = \frac{1}{\left(\int_{\Omega} u^{\frac{2n}{n-2}} dx \right)^{\frac{n}{n-2}}}, \quad u \in \Sigma$$

be the Euler Lagrange functional associated to Yamabe problem on Ω . Let

$$S = \frac{1}{\left(\int_{\mathbb{R}^n} \delta_{a,\lambda}^{\frac{2n}{n-2}} dx \right)^{\frac{n}{n-2}}}$$

be the best Sobolev constant. S does not depend on a and λ . It is known that

$$S = \inf_{u \in \Sigma} J_1(u)$$

and that the infimum is not achieved, see [9].

For $c \in \mathbb{R}$ and for any function f on Σ , we define

$$f^c = \{u \in \Sigma, \text{ s.t. } f(u) \leq c\}.$$

It is easy to see that if $|K - 1|_{L^\infty(\Omega)} \leq \varepsilon_0$ for ε_0 small enough, we have

$$J^{S+\frac{S}{4}} \subset J_1^{S+\frac{S}{2}} \subset J^{S+\frac{3S}{4}}. \quad (5.2)$$

This is due to the fact that $J(u) = J_1(u)(1 + O(|\varepsilon_0|))$.

Now let $(y_1, \dots, y_q)_\infty$ be a critical point at infinity of q masses. It is known that the level of J at $(y_1, \dots, y_q)_\infty$ is given by $S \left(\sum_{k=1}^q \frac{1}{K(y_k)^{(n-2)/2}} \right)^{2/n}$, see [8]. Hence goes to qS when ε_0 is close to zero. Therefore, for $|\varepsilon_0|$ small enough, we have:

All critical points at infinity of J of q -masses, $q \geq 2$ are above $S + \frac{3}{4}S$, $(*)$

and

all critical points at infinity of J of one masse are below $S + \frac{S}{4}$. $(**)$

Therefore,

$$\text{the functional } J \text{ has no critical points at infinity in } J_{(\tilde{S}_n+\eta)}^{(\tilde{S}_n+3\eta)}. \quad (5.3)$$

To prove the existence result, we argue by contradiction and we assume that J has no critical points. It follows from (5.3) that

$$J^{\tilde{S}_n+3\eta} \simeq J^{\tilde{S}_n+\eta},$$

where \simeq denotes retracts by deformation. Thus by (5.2), we derive that

$$J_1^{\tilde{S}_n+2\eta} \simeq J^{\tilde{S}_n+\eta}. \quad (5.4)$$

Now we use the gradient flow of $(-\partial J)$ to deform $J^{\tilde{S}_n+\eta}$. As mentioned above, the only critical points at infinity of J under the level $\tilde{S}_n + \eta$ are $(y)_\infty$, $y \in \mathcal{K}_{< n-4}^+ \cup \mathcal{K}_{\geq n-4}$. Thus

$$J^{\tilde{S}_n+\eta} \simeq \bigcup_{y \in \mathcal{K}_{< n-4}^+ \cup \mathcal{K}_{\geq n-4}} W_u^\infty(y). \quad (5.5)$$

We apply now the Euler-Poincaré characteristic of both sides of (5.5), we get

$$\chi(J^{\tilde{S}_n+\eta}) = \sum_{y \in \mathcal{K}_{< n-4}^+ \cup \mathcal{K}_{\geq n-4}} (-1)^{n-\tilde{i}(y)}.$$

Thus by (5.4), we obtain

$$\chi(J_1^{\tilde{S}_n+2\eta}) = \sum_{y \in \mathcal{K}_{< n-4}^+ \cup \mathcal{K}_{\geq n-4}} (-1)^{n-\tilde{i}(y)}. \quad (5.6)$$

It is known that $J_1^{\tilde{S}_n+2n}$ and Ω has the same homotopy type. See ([5] remark 5). Therefore, from (5.6) we get

$$\chi(\Omega) = \sum_{y \in \mathcal{K}_{<n-4}^+ \cup \mathcal{K}_{\geq n-4}} (-1)^{n-\tilde{i}(y)}. \quad (5.7)$$

Such equality contradicts the assumption of Theorem 1.4. This complete the proof of Theorem 1.4.

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